RECONSTRUCTION OF THE TEMPERATURE FIELD IN A SOLID FROM A LIMITED NUMBER OF MEASUREMENTS

O. V. Minin and N. A. Yaryshev

UDC 536.21.083

We have obtained extrapolation formulas for calculating the temperatures, temperature gradients, and heat fluxes in a solid heated by a varying thermal flux from the results of temperature measurements at two points. The applicability limits and errors of the proposed formulas have been investigated.

In investigating nonsteady-state thermal conductivity in a solid, it is often necessary to determine the temperature field from the results obtained in measuring the temperatures at a limited number of points.

The problem of reconstruction of the temperature at the boundary of a solid from the experimentally determined temperatures measured at two points within it were considered by Tikhonov and Glasko [1]. An algorithm for determination of the boundary temperature was devised on the basis of solution of the first boundary problem for thermal conductivity, as incorrectly formulated by the regularization method [2].

We previously [3] gave a nonlinear extrapolation function that can be used to calculate the surface temperature of a body during heating at the boundary with a constant heat flux.

However, the possibility of temperature reconstruction from a minimum number of experimental data over a broader range of variation in the coordinate and time was not investigated. This is a pressing problem for certain areas of thermoelasticity, thermometry, and thermophysical measurement.

In many cases, the action of external energy on a body is such as to lead to a monotonic change in the heat flux passing through the body surface with time.

We will consider the heating of a semibounded homogeneous body, whose thermophysical parameters are independent of the temperature, by a heat flux that varies with time by the rule $q(\tau) = q_0 + b\tau$, where $q_0 = \mathrm{const}$ and $b > 0$. Since we assume that the heating process is not accompanied by phase transitions or chemical reactions, the problem reduces to integration of the Fourier equation.

An exact solution to the problem is easily obtained on the basis of the Duhamel theorem, since the expression for the temperature in a solid with a constant heat flux at its boundary is well known [4]:

$$\vartheta(x,\ \tau) = \frac{q_0}{\lambda}\sqrt{a\tau}\left(2\,\mathrm{ierfc}\,\frac{x}{2\sqrt{a\tau}} + 8\,\frac{b\tau}{q_0}\,\mathrm{i}^3\,\mathrm{erfc}\,\frac{x}{2\sqrt{a\tau}}\right). \tag{1}$$

However, the relationship obtained is inconvenient for extrapolation of the temperature beyond the measurement points. We will construct an approximate expression for the temperature profile in the heated body. The initial thermal conductivity equation is written in the form

$$L(\vartheta) = a\nabla^2\vartheta - \frac{\partial\vartheta}{\partial\tau} = 0. \tag{2}$$

We use the concept of the depth of penetration of the heated zone $\delta(\tau)$, as was done in other studies [5, 6]. The boundary conditions of the problem then take the form

$$-\lambda\,\frac{\partial\vartheta(0,\ \tau)}{\partial x} = q_0 + b\tau;\quad \frac{\partial\vartheta(\delta(\tau),\ \tau)}{\partial x} = 0;\qquad \vartheta(\delta(\tau),\ \tau) = 0;\quad \vartheta(x,\ 0) = 0. \tag{3}$$

Institute of Precision Mechanics and Optics, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 3, pp. 553-558, September, 1969. Original article submitted October 7, 1968.

The Kantorovich method, in combination with the moment method [7, 8], is used to integrate Eq. (2) with the conditions in Eq. (3). An approximate solution, taking into account the null initial conditions, is sought in the form

$$\tilde{\vartheta}(x, \tau) = \sum_{i=1}^{m} \eta_i(\tau)\varphi_i(x, \tau), \tag{4}$$

where $\varphi_i(x, \tau)$ are linearly independent coordinate functions selected a priori and $\eta_i(\tau)$ are functions that can be determined from the corresponding system of ordinary differential equations.

Limiting ourselves to one term of the sum in Eq. (4), we write the first approximation of the problem in the form of the following power function:

$$\tilde{\vartheta}(x, \tau) = \frac{q(\tau)}{\lambda}\delta(\tau)\frac{1}{n}\left[1 - \frac{x}{\delta(\tau)}\right]^n, \tag{5}$$

where n has not yet been defined.

It is easy to verify that Eq. (5) satisfies the boundary conditions in Eq. (3). The heated-layer function $\delta(\tau)$ can be found from the noncorrelation orthogonality condition $L(\tilde{\vartheta})$ of a certain weighted function ζ:

$$\int_0^\delta L(\tilde{\vartheta})\zeta dx = 0. \tag{6}$$

If we assume $\zeta = 1$ in Eq. (6), the integral-balance method applies [5, 6]; $\zeta = \varphi(x)$ yields the Galerkin method and $\zeta = \partial\varphi/\partial\delta$ yields the method employed by Kogan [7].

We select the function ζ in the form

$$\zeta = \frac{\partial\varphi}{\partial\delta} = \left(1 - \frac{x}{\delta}\right)^{n-1}\frac{x}{\delta^2}. \tag{7}$$

Substituting Eqs. (5) and (7) into Eq. (6) and integrating, we obtain the following ordinary differential equation with respect to δ:

$$\frac{d\delta^2}{d\tau} + \frac{4n-2}{4n-1}\,\frac{b}{q_0 + b\tau}\,\delta^2 = an^2\,\frac{4n+2}{4n=1}, \tag{8}$$

where a is the coefficient of thermal conductivity of the material.

Taking into account the initial condition $\delta(\tau)|_{\tau=0} = 0$, we obtain an expression for the heated zone in the following form:

$$\delta(\tau) = n\sqrt{\frac{4n+2}{8n-3}a\tau\left(1 + \frac{1}{\alpha}\right)\left[1 - \left(\frac{1}{1+\alpha}\right)^{\frac{8n-3}{4n-1}}\right]}, \tag{9}$$

where

$$\alpha = \frac{b\tau}{q_0}.$$

It specially follows from Eq. (9) that, with $q(\tau) = q_0$ and $q(\tau) = b\tau$, the heated zone in the body can be calculated from the corresponding relationships

$$\delta(\tau) = n\sqrt{\frac{4n+2}{4n-1}a\tau};$$
$$\delta(\tau) = n\sqrt{\frac{4n+2}{8n-3}a\tau}. \tag{10}$$

When Eq. (9) [or Eq. (10)] is substituted into Eq. (5), we obtain an approximation of the temperature field with an accuracy to the parameter n. The approximate solution to the problem of heating with a varying heat flux can be used to construct practical extrapolation formulas.

Let temperature measurements be made at the points x_1 and x_2; then, in accordance with Eq. (5), we can construct three equations for $x = x_1$, $x = x_2$, and $x = x$. Joint solution of the system for $\tilde{\vartheta}(x, \tau)$ gives the relationship among the temperatures at these points:

$$\tilde{\vartheta}(x,\ \tau)=\tilde{\vartheta}(x_1,\ \tau)\left[\frac{x_2-x}{x_2-x_1}-\frac{x_1-x}{x_2-x_1}\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}\right]^n. \tag{11}$$

The applicability limits for Eq. (11) with respect to x and τ, for a given accuracy can be determined from comparison of the precise (1) and approximate (5) solutions over the space–time range $Z = x/2\sqrt{a\tau}$. Since Eq. (5) has an accuracy to the parameter n, the "best" value of the latter must be regarded as the integer that yields the least point-by-point deviation of the approximate solution of Eq. (1) from its exact solution over the greatest range of variation in Z. This requires that the precise solution $\vartheta(x,\ \tau)$ and approximate solution $\tilde{\vartheta}(x,\ \tau)$ coincide at the boundary of the body. As a result, we obtain the following equation for determination of the power n:

$$\frac{4n+2}{8n-3}\left[1-\left(\frac{1}{1+\alpha}\right)^{\frac{8n-3}{4n-1}}\right]=\frac{\alpha(1.128+0.752\,\alpha)^2}{(1+\alpha)^3}. \tag{12}$$

For the special cases considered above, in which heat fluxes $q(\tau) = q_0$ and $q(\tau) = b\tau$ act on the body, we obtain power values n of 2.93 and 7.04 respectively; rounding these off to the integers 3 and 7 introduces no error. Analysis shows that the region of Z in which the temperature and gradient values coincide with an error of no more than 3% (at the end of the interval) in these cases yields the corresponding inequalities

$$0 \leqslant Z \leqslant 0.5,\quad 0 \leqslant Z \leqslant 0.8. \tag{13}$$

With known values for the coordinate x, we determine the time after which the approximations $\vartheta(x,\ \tau)$ and $\partial\vartheta/\partial x$ are satisfied with given accuracy at the point in question and, conversely, the region [0, x) where the exact and approximate values for the temperature or gradient "coincide" becomes known at a given value of τ.

As follows from Eq. (12), the value of n depends on $\alpha = b\tau/q_0$ in the case $q(\tau) = q_0 + b\tau$. This relationship is shown in Fig. 1.

Figure 2 shows the relative error $\Delta = [\tilde{\vartheta}(x,\ \tau) - \vartheta(x,\ \tau)]/\vartheta(x,\ \tau)$ in the approximation formula in Eq. (5) as a function of Z and α with n = 7.

It can be seen from these graphs that the value of n for $\alpha \geq 1$ can be assumed to equal 7, i.e., at a certain point in time during thermal evolution, the initial level of the incoming heat flux no longer affects the form of the temperature distribution and we have the case $q(\tau) = b\tau$, with the accuracy estimates valid for it.

With $0 < \alpha \leq 1$, the value of n is also 7, but, as can be seen from Fig. 2, the exactness of the agreement of $\tilde{\vartheta}$ and ϑ decreases to 5% in this case for the same range of variation in Z ($0 \leq Z \leq 0.8$).

The extrapolation function in Eq. (11) thus enables us to reconstruct the temperatures in the body from the two measured temperatures; we need not know the impinging heat flux, the rate at which it changes with time, or the thermophysical constants of the material, i.e., those quantities that are usually unknown under experimental conditions.

Assuming $x_2 = 2x_1$ in Eq. (11), which can occur in practice, we specifically find that the following simple relationship obtains for calculation of the temperature at the boundary of the body with x = 0:

$$\tilde{\vartheta}(0,\ \tau)=\tilde{\vartheta}_1\left[2-\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}\right]^n. \tag{14}$$

We can obtain a linear extrapolation formula from Eq. (14) as a special case [1]. For this purpose, we must assume n = 1.

Equation (11) readily yields a relationship for reconstruction of the temperature gradient from the temperatures measured at two points:

$$\frac{\partial\tilde{\vartheta}}{\partial x}=-n\tilde{\vartheta}_1\left[\frac{x_2-x}{x_2-x_1}-\frac{x_1-x}{x_2-x_1}\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}\right]^{n-1}\left[\frac{1}{x_2-x_1}-\frac{1}{x_2-x_1}\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}\right]. \tag{15}$$

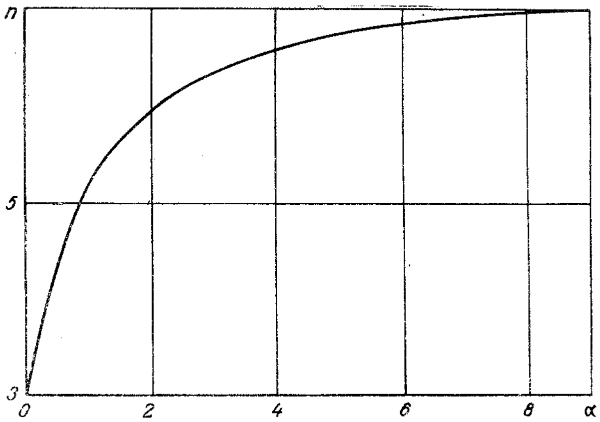

Fig. 1

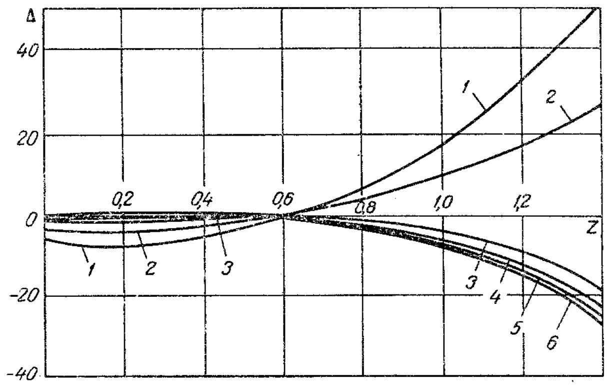

Fig. 2

Fig. 1. Power n as a function of α.

Fig. 2. Relative error in approximate solution of Eq. (5) as a function of Z and α: 1) $\alpha = 0$; 2) 0.5; 3) 2; 4) 4; 5) 8; 6) 10.

By making simple transformations, the gradient at the boundary of the body can be expressed in terms of the surface temperature:

$$\frac{\partial\tilde{\vartheta}(0,\ \tau)}{\partial x} = -\tilde{\vartheta}_1 \frac{n}{x_1}\left[2-\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}\right]^n \left[1-\frac{1}{2-\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}}\right] = -\frac{n}{x_1}\tilde{\vartheta}_0\left[1-\frac{1}{2-\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}}\right]. \tag{16}$$

If we know the coefficient of thermal conductivity of the material λ, the thermal flux impinging on the body is

$$\tilde{q}_0(0,\ \tau) = \lambda\frac{n}{x_1}\tilde{\vartheta}_0\left[1-\frac{1}{2-\left(\frac{\tilde{\vartheta}_2}{\tilde{\vartheta}_1}\right)^{\frac{1}{n}}}\right]. \tag{17}$$

In order to determine $\tilde{\vartheta}_0$, $\partial\tilde{\vartheta}/\partial x$, and $\tilde{q}_0(0,\ \tau)$ with the requisite accuracy (no more than 5%), it is necessary that the appropriate condition from Eq. (13) be satisfied, where x is assumed to equal x_2.

The inequalities in Eq. (13) define the lower limit of applicability of the extrapolation formulas obtained. For a linear process, the upper limit is determined by the point at which the surface begins to melt.

The approximate solution given by Eq. (5) enables us to obtain still another extrapolation relationship for determination of the temperature at any point x in the region $0 < x < \sqrt{a\tau_*}$ at the instant $\tau > \tau_*$ from the temperatures measured at the times τ_1 and τ_2: $\tau_2 > \tau_1 > \tau_*$. Solving the system of three equations in the form of Eq. (5) for, e.g., $\tau = \tau$, $\tau = \tau_1$, and $\tau = \tau_2$ at $x = x_1$, we obtain the following formula:

$$\vartheta(x,\ \tau) = \vartheta(x_1,\ \tau)\sqrt{\frac{\tau}{\tau_1}}\left\{1-\frac{1-\sqrt{\frac{\tau_1}{\tau}}}{1-\sqrt{\frac{\tau_1}{\tau_2}}}+\left[\frac{\vartheta(x_1,\ \tau_2)}{\vartheta(x_1,\ \tau_1)}\sqrt{\frac{\tau_1}{\tau_2}}\right]^{\frac{1}{n}}\frac{1-\sqrt{\frac{\tau_1}{\tau}}}{1-\sqrt{\frac{\tau_1}{\tau_2}}}\right\}^n. \tag{18}$$

If the ratio $\tau_2/\tau_1 < 3$, Eq. (18) can be simplified to

$$\vartheta(x,\ \tau) = \vartheta(x_1,\ \tau)\sqrt{\frac{\tau}{\tau_1}}\left[1-\frac{1-\sqrt{\frac{\tau_1}{\tau}}}{1-\sqrt{\frac{\tau_1}{\tau_2}}}\left(1-\frac{\vartheta_{12}}{\vartheta_{11}}\right)^{\frac{1}{n}}\right]^n, \tag{19}$$

where

$$\vartheta_{11} = \vartheta(x_1, \tau_1); \; \vartheta_{12} = \vartheta(x_1, \tau_2).$$

LITERATURE CITED

1. A. N. Tikhonov and V. B. Glasko, ZhVMiMF, 7, No. 4 (1967).
2. A. N. Tikhonov, Dokl. Akad. Nauk SSSR, 153, No. 1 (1963).
3. N. A. Yaryshev, Theoretical Principles of the Measurement of Nonsteady-State Temperatures [in Russian], Energiya, Leningrad (1967).
4. A. V. Lykov, Theory of Thermal Conductivity [in Russian], Vysshaya Shkola, Moscow (1967).
5. G. N. Barenblatt, Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, No. 9 (1954).
6. T. R. Gudmen, in: Problems of Heat Transfer [in Russian], P. L. Kirillova (editor), Atomizdat (1967).
7. M. G. Kogan, Inzh.-Fiz. Zh., 12, No. 1 (1967).
8. A. V. Kantorovich and V. I. Krylov, Approximate Methods for Higher Analysis [in Russian], Fizmatgiz (1962).